Discipline: **Physics**
Subject: **Electromagnetic Theory**
Unit 24:
Lesson/ Module: **Energy Loss in Collisions - I**

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Learning Objectives:

From this module students may get to know about the following:

1. *An overview of the problem of energy loss by a charged particle in passing through matter.*
2. *Energy transfer to a free atomic electron by a fast moving charged particle.*
3. *Limitations of the formula in the small distance and large distance limits and the corresponding corrections.*
4. *Result for the case of harmonically bound charge.*

24. Energy Loss in Collisions - I

24.1 Introduction

In this module we begin our study of collisions between fast moving charged particles. In a collision between charged particles there is exchange of energy between the particles and there is accompanying deflections. Our aim is to find loss of energy by a charged particle moving through a medium. This is an interesting system from many points of view. Historically it played an important role in resolving the question of the structure of matter by Rutherford. Energy loss is an important phenomenon in particle physics, nuclear engineering and solid state physics in the study of properties of materials and radiation damage to materials.

A fast moving charged particle when it passes through matter makes collisions both with the atomic electrons and the nuclei. The resulting energy loss and deflection of the particle by the electrons and the nuclei depends on its mass. If the particle is considerably heavier than electrons, its collision with the electrons and with the nuclei have different consequences. The electron being lighter than the incident particle, is able to absorb appreciable amount of energy but does not cause much of a deflection of the incident particle. On the other hand the massive nuclei (it is assumed that the incident particle is much lighter than the nuclei, say a pion, a kaon, a muon, a proton or an alpha particle etc) can take up very little energy but because of much larger charge, can cause scattering of the incident particle. Thus loss of energy of the incident particle occurs almost exclusively from collisions with electrons and the deflection of the particle from collisions with the nuclei. The scattering is confined, by and large, to small angles, so that the particle keeps a more or less straight path until it loses most of its energy and comes near the end of its range before it is finally absorbed.

If the incident particle is an electron it loses energy as well as suffers scattering in collisions with atomic electrons. As a result their path in matter is much less straight; they have a rather short range after which they simply diffuse in the surrounding material.

The subject of energy loss and scattering is of great practical interest and is a rather technical subject. For obtaining accurate results a proper quantum-mechanical treatment is required. However we are more interested here in the physical ideas involved rather than the detailed tables of ranges of various subatomic particles in various media. All the essential features can be understood from a classical treatment and the order of magnitude of the quantum-mechanical effects can be obtained from the use of uncertainty principle.

24.2 Energy transfer in Coulomb collisions

We begin by considering the problem of energy transfer to an atomic electron by a fast moving charged particle assumed to be much heavier than the electron. Let the mass of the particle be M and charge ze. If the speed of the particle is large compared to the characteristic speed of the electron in its orbit, the electron can be considered as essentially free as well as at rest. We further assume that the momentum transfer Δp to the electron is sufficiently small so that the incident particle is essentially undeflected from its straight-line path. The recoiling electron also does not move appreciably from its position of equilibrium. Thus all we need to calculate is the impulse delivered to the electron by the electric field of the incident particle. Since the electron is essentially at rest, the effect of the magnetic field can be ignored. As we saw in our study of the theory of relativity, the magnetic field produced by a moving charge is of the order of u/c^2

compared to the electric field, where *u* is the speed of the moving particle. Thus the force due to the magnetic field is of the order uv/c^2 compared to that due to the electric field and is negligible even for relativistically moving particle.

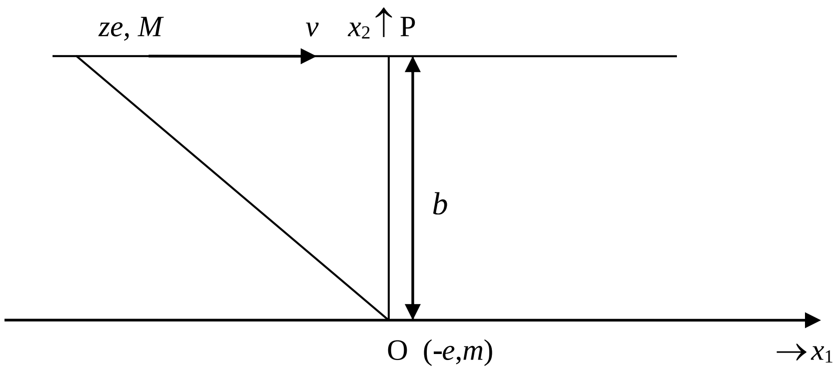


Fig. 24.1

The collision is depicted in the figure above. The incident particle has mass *M*, charge *Ze*, speed *v* and energy $E = \gamma Mc^2; \gamma = (1-\beta^2)^{-1/2}; \beta = v/c$. For the incident particle we have used the relativistic formula for energy. The electron which is positioned at O, has mass *m* and charge $-e$. The distance OP is the impact parameter, the distance of closest approach of the particle to the electron. In terms of these parameters, the fields produced by the incident particle at the position of the electron are given by

$$
\begin{aligned}
E_x &= -\frac{1}{4\pi\varepsilon_0}\frac{ze\gamma vt}{[b^2+(\gamma vt)^2]^{3/2}}, \\
E_y &= \frac{1}{4\pi\varepsilon_0}\frac{\gamma zeb}{[b^2+(\gamma vt)^2]^{3/2}} \qquad (1) \\
B_z &= \frac{1}{4\pi\varepsilon_0 c}\frac{\gamma zeb\beta}{[b^2+(\gamma vt)^2]^{3/2}} = \beta E_y / c
\end{aligned}
$$

[See module on Theory of relativity for details.]

Of the *x* and *y* components of the electric field, the *x* component being an odd function of time its time integral is zero. The momentum impulse is, therefore, in the transverse direction and has the magnitude

$$\Delta p = \int_{-\infty}^{\infty} eE_y(t)dt = \frac{1}{4\pi\varepsilon_0}\frac{2ze^2}{bv}. \qquad (2)$$

Interestingly, the impulse, Δp, is independent of γ. As γ increases, the peak (which is $t = 0$) field increases in magnitude, but the duration for which it is significant decreases in inverse proportion and velocity times the time integral is independent of γ. The energy transferred to the electron, as a function of the impact parameter, is then

$$\Delta W(b) = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{2z^2e^4}{mv^2}\frac{1}{b^2}. \qquad (3)$$

The energy loss is directly proportional to the square of the charge of the target particle and inversely proportional to its mass. Thus for the nucleus of charge *Ze* the result would have multiplied by a factor of $Z^2/(1840A)$. However there are *Z* electrons per nucleus so that overall energy loss to the nucleus is down by a factor of $Z/(1840A)$. In other words, it is completely negligible.

If, as we have in any case assumed, $\Delta p << p$, then the angular deflection of the incident particle is given by $\theta = \frac{\Delta p}{p}$. Thus for small deflections

$$\theta = \frac{1}{4\pi\varepsilon_0}\frac{2ze^2}{pvb}. \qquad (4)$$

This result agrees with the exact expression for the Rutherford scattering of a non-relativistic charged particle for small angles:

$$2\tan\left(\frac{\theta}{2}\right) = \frac{1}{4\pi\varepsilon_0}\frac{2ze^2}{pvb} \qquad (5)$$

24.2.1 Small distance limit

The energy transfer given by equation (3) has several interesting features. It depends on the charge and velocity of the incident particle but not on its mass. It varies as inverse square of the impact parameter so that close collisions involve very large energy transfers. However as *b* tends to zero the formula gives infinite energy transfer which is obviously meaningless. The formula is actually valid only for large values of *b*. The energy transfer obviously cannot exceed the maximum possible energy transfer that takes place when the collision is head-on. The maximum energy transfer can be easily calculated from law of conservation of energy and momentum. If a particle of mass *m* and velocity *v* hits head-on a particle of mass *M* initially at rest, the amount of energy transferred can be calculated to be

$$\Delta W_{\max} = \frac{2m\gamma^2v^2}{1+2(M/m)\gamma+(M/m)^2}. \qquad (6)$$

If the mass of the incident particle is large compared to that of the initially stationary particle, as is the case under consideration, $M<<m$, the above expression reduces to

$$\Delta W_{\max} \approx 2m\gamma^2v^2. \qquad (7)$$

Equating this to the value of ΔW from equation (3), we obtain the value $b_{\min}$ of *b* up to which we can expect formula (3) to be valid as

$$b_{\min} = \frac{1}{4\pi\varepsilon_0}\frac{ze^2}{\gamma mv^2}\,. \tag{8}$$

We can argue in an alternative way to obtain this limiting value of $b_{\min}$. In deriving equation (3) we had assumed that the initially stationary electron does not move appreciable during the collision. As long as the distance it actually moves is small compared to b, equation (3) should be approximately valid. The distance traversed by the electron can be estimated in the following way. The momentum of the electron changes from zero to Δp, so the average speed of the electron is $\Delta p/2m$. From equation (1) we see that the time for which the field E_y of the incoming particle is appreciable is

$$\Delta t \approx \frac{b}{\gamma v}\,. \tag{9}$$

Thus the distance traveled by the electron during the collision is of the order of $d \approx \frac{\Delta p}{2m}\Delta t$. Substituting for Δp from equation (2), we have

$$d = \frac{1}{4\pi\varepsilon_0}\frac{ze^2}{\gamma mv^2} = b_{\min} \tag{10}$$

.

Note that these formulae are not correct for incident particles with too high energy because of the other factors in the denominator in equation (6). For muons, $(M/m) \approx 207$, and the denominator must be taken into account if the energy is comparable to or greater than 44 GeV. For protons this energy comes to about 340 GeV. For equal mass particles, $\Delta W_{\max} = (\gamma - 1)mc^2$.

24.2.2 Large distance limit

The result for energy transfer given by equation (3) is approximate for large distances as well. This is because of the binding of the electrons in the atom which has been neglected since we have assumed the electrons to be free. As long as the collision time given by equation (9) is short compared to the orbital period of the electron in its orbit, the collision is sudden enough and the electron can be treated as free. On the other hand if the orbital period is very short compared to Δt, the electron will make many rounds during the time the incident particle is passing by. The atom then responds adiabatically – it stretches slowly during the encounter and then returns to normal, with no appreciable transfer of energy. If ω is the characteristic frequency of the electron, the dividing line between the two extremes is given by $\omega\Delta t(b_{\max}) \approx 1$, or

$$b_{\max} = \frac{\gamma v}{\omega} \tag{11}$$

Beyond this no significant energy transfer is possible.**[See figure 13.2 from Jackson Edition 2]**

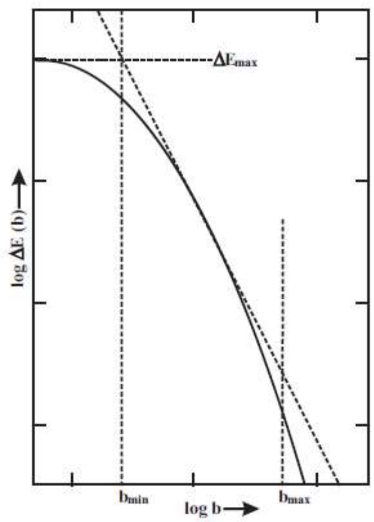


Fig: Energy transfer as a function of impact parameter

The figure shows the graph between the impact parameter *b* and energy loss $\Delta E(b)$ on the log scale. The dotted curve depicts the approximate form, equation (3), while the solid curve represents the "correct" result taking into account the deviations both at low *b* and high *b*. The approximate result is reasonably accurate at intermediate values of *b*: $b_{min} < b < b_{max}$, deviating more and more from it as *b* goes beyond b_{max} or below b_{min}.

14.2.3 The energy loss per unit distance

What we have calculated so far is the energy transferred by the moving particle to a single electron at a fixed impact parameter *b*. However as the fast moving particle passes through matter, it sees electrons at various distances from its path. We are interested in the energy loss by the particle per unit distance of its travel. If there are *N* atoms per unit volume of the matter and each atom has *Z* electrons, the number of electrons located at impact parameters between *b* and (*b* + *db*) in a thickness *dx* of matter is

$$dn = NZ\, b(2\pi db)(dx) \ . \tag{12}$$

Since the energy loss due to collision with a single electron is $\Delta E(b)$, the total energy loss per unit distance by the incident particle due to collisions with all the particles is

$$\frac{dW}{dx} = 2\pi NZ \int \Delta E(b) b\, db \, . \tag{13}$$

The integration over *b* is to be done from zero to infinity. However in view of the behavior of $\Delta E(b)$ as shown in the diagram, we may use the approximate formula (3) and integrate from b_{min} to b_{max}. In this process we overestimate the contribution from b_{min} to b_{max} by a small margin, but ignore a small contribution from outside this range. As a result of this integration

$$\frac{dW}{dx} = 4\pi NZ\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{z^2e^4}{mv^2}\int_{b_{\min}}^{b_{\max}} \frac{1}{b^2}\, b\, db \tag{14}$$

or

$$\frac{dW}{dx} = 4\pi NZ\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{z^2e^4}{mv^2}\ln B \tag{15}$$

where

$$B = \frac{b_{\max}}{b_{\min}} = \frac{\gamma^2 mv^2}{ze^2\omega} \tag{16}$$

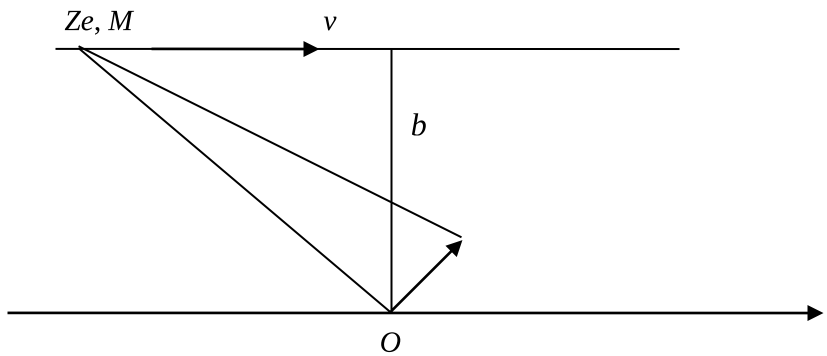
The lower limit can be tackled more accurately. A proper treatment of the scattering process for any impact parameter b yields for the energy transfer the expression

$$\Delta W(b) = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{2z^2e^4}{mv^2}\frac{1}{b^2 + b_{\min}{}^2} \tag{17}$$

where $b_{\min}$ is given by equation (8). Using this expression for $\Delta W(b)$ in equation (13) and integrating from zero to $b_{\max}$ yields exactly the same result, viz.; equation (15). However the cutoff at $b_{\max}$ is only approximate. Consequently, the expression for B is uncertain to within a small factor. Since it is logarithm of B that appears in the expression for energy loss, this factor is not of much consequence.

14.3 The case of harmonically bound charge

We now consider the more realistic case of the electron being harmonically bound rather than being free. This serves as a simple but more realistic model of the energy lost by a charged particle passing through matter. Thus we have a charged particle of charge ze, velocity v and mass M which is much greater than the mass m of the bound charge of value $-e$. As before we will assume that the incoming particle being massive deviates only slightly from its path which can thus be taken to be a straight line. The impact parameter, the distance of closest approach to the bound charge is b. [See figure below.]



The effect of binding is obviously significant only at large impact parameters. Thus we can assume that the energy transfer is not very large. As a result, the motion of the bound charge is also non-relativistic throughout, and its amplitude of oscillations about the origin remains small compared to the impact parameter. Under these circumstances, the effect of magnetic part of the Lorentz force can be neglected and only the force due to electric field need be included. Furthermore, the spatial variation of the field over the position of the particle may be neglected, and its value may be taken to be that at the origin, the position of equilibrium of the charged particle. This is sometimes called *the dipole approximation*.

With these approximations, the equation of motion for the harmonically bound charge may be written as

$$\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + {\omega_0}^2\vec{x} = -\frac{e}{m}\vec{E}(t) \tag{18}$$

Here $\vec{E}(t)$ is the electric field at the origin due to the moving charged particle; its components are given by equation (1). ω_0 is the characteristic frequency of oscillation of the bound charge and Γ is the usual damping factor.

One way to solve this inhomogeneous differential equation is to take the Fourier transform of both $\vec{E}(t)$ and $\vec{x}(t)$. So let

$$\vec{x}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{x}(\omega)e^{-i\omega t}d\omega \tag{19}$$

$$\vec{E}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(\omega)e^{-i\omega t}d\omega \tag{20}$$

Since $\vec{x}(t)$ [and also $\vec{E}(t)$] is real, on taking the complex conjugate of equation (19) we obtain

$$\vec{x}(t) = \vec{x}^*(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{x}^*(\omega)e^{i\omega t}d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{x}(-\omega)e^{i\omega t}d\omega$$

Or

$$\vec{x}(-\omega) = \vec{x}^*(\omega) \tag{21}$$

Similarly

$$\vec{E}(-\omega) = \vec{E}^*(\omega) \tag{22}$$

On substituting the Fourier integral forms of $\vec{E}(t)$ and $\vec{x}(t)$, equations (19) and (20), into the equation of motion (18), we obtain

$$\vec{x}(\omega) = -\frac{e}{m}\frac{\vec{E}(\omega)}{\omega_0^{\ 2} - i\omega\Gamma - \omega^2} \qquad (23)$$

Given $\vec{E}(t)$, $\vec{E}(\omega)$ can be determined from the inverse Fourier transform of equation (21):

$$\vec{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}(t)e^{i\omega t}dt \qquad (24)$$

Substitution of $\vec{E}(\omega)$ into equation (23) determines $\vec{x}(\omega)$, and then substitution of $\vec{x}(\omega)$ into equation (19) determines $\vec{x}(t)$. All this, of course, if the various integrals can be performed analytically.

However, what we are really interested in is the energy transfer to the bound charge in the collision rather than its detailed motion. The energy transfer can be found by considering the work done by the incident particle on the bound one. The rate of doing work is given by

$$\frac{dW}{dt} = \vec{F}.\frac{d\vec{x}}{dt} = -e\vec{E}.\vec{v} \qquad (25)$$

The total work done by the particle passing by on the bound charge is thus

$$\Delta W = \int_{-\infty}^{\infty} \frac{dW}{dt}dt = -e\int_{-\infty}^{\infty} \vec{v}.\vec{E}\,dt \qquad (26)$$

In the dipole approximation that we have made, $\vec{E}(t)$ is the electric field of the incident particle at the position of equilibrium of the bound charge.

To perform the integral in equation (26), we use the Fourier representations of $\vec{E}(t)$ and $\vec{x}(t)$, equations (19) and (20). This gives

$$\Delta W = -e\int_{-\infty}^{\infty} dt \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-i\omega)\vec{x}(\omega).e^{-i\omega t}d\omega \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}(\omega')e^{-i\omega' t}d\omega'$$
$$= -e\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (-i\omega)\vec{x}(\omega).\vec{E}(\omega')d\omega\, d\omega' \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega+\omega')t}dt$$

Now we use the Fourier representation of the delta function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}dt$$

and get

$$\Delta W = -e\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-i\omega)\vec{x}(\omega).\vec{E}(\omega')d\omega\, d\omega'\delta(\omega+\omega')$$
$$= e\int_{-\infty}^{\infty}(i\omega)\vec{x}(\omega).\vec{E}(-\omega)d\omega \qquad (27)$$

Further simplification can be achieved on using equations (21) and (22) which connect the negative and positive frequency parts of $\vec{E}(\omega)$ and $\vec{x}(\omega)$. Splitting integral (26) into two parts, from $(-\infty,0)$ and $(0,\infty)$, and using equations (21) and (22) in the first part, we obtain

$$\Delta W = 2e\,\mathrm{Re}\int_{0}^{\infty}(i\omega)\vec{x}(\omega).\vec{E}^{*}(\omega)d\omega \qquad (28)$$

Now using equation (23) for $\vec{x}(\omega)$ in terms of $\vec{E}(\omega)$, we obtain

$$\Delta W = -2e\,\mathrm{Re}\int_{0}^{\infty}(i\omega)\frac{e}{m}\frac{|E(\omega)|^2}{\omega_0{}^2 - i\omega\Gamma - \omega^2}d\omega$$

On rationalizing the denominator in the integrand we obtain

$$\Delta W = \frac{e^2}{m}\int_{0}^{\infty}\frac{2\omega^2\Gamma\,|E(\omega)|^2}{(\omega_0{}^2-\omega^2)^2+\omega^2\Gamma^2}d\omega \qquad (29)$$

We continue to make further approximations to get an idea of the effect of binding on the energy transfer. For small Γ the integrand peaks sharply around $\omega=\omega_0$ in an approximately Lorentzian line shape. (The Lorentzian function is the singly peaked function given by $L(x)=\dfrac{1}{\pi}\dfrac{\Gamma/2}{(x-x_0)^2+(\Gamma/2)^2}$, where x_0 is the centre and Γ is a parameter specifying the width. The Lorentzian function is defined so that $\int_{-\infty}^{\infty}L(x)dx=1$.) Since the integrand is sharply peaked around $\omega=\omega_0$, we can replace the electric field $\vec{E}(\omega)$ by $\vec{E}(\omega_0)$, its value at the maximum of the sharp peak. Then equation (29) becomes

$$\Delta W = \frac{2e^2}{m}|E(\omega_0)|^2\int_{0}^{\infty}\frac{\omega^2\Gamma}{(\omega_0{}^2-\omega^2)^2+\omega^2\Gamma^2}d\omega$$

By a change of variable, $\dfrac{\omega}{\Gamma}=x$, the integral becomes

$$\int_{0}^{\infty}\frac{x^2dx}{\left(\dfrac{\omega_0{}^2}{\Gamma^2}-x^2\right)^2+x^2}$$

This integral has the value $\pi/2$ independent of ω_0/Γ. Thus finally we have

$$\Delta W = \frac{\pi e^2}{m} |\vec{E}(\omega_0)|^2 .$$

(30)

This is a very general result for energy transfer to a nonrelativistic oscillator by an external field.. In the present case the field is produced by a moving charged particle.

For a particle with charge *ze*, velocity *v*, and impact parameter *b* (see figure 24.1), the nonzero electromagnetic fields at the origin *O* are given by equation (1). Then $\vec{E}(\omega_0)$ is given by the inverse Fourier transform of equation (20):

$$\vec{E}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(t) e^{i\omega t} dt \qquad (31)$$

Using equation (1) for the *y* component of $\vec{E}(t)$ we obtain

$$E_y(\omega) = \frac{1}{4\pi\varepsilon_0} \frac{ze\gamma b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} dt \qquad (32)$$

Now on changing the integration variable to $u = \gamma vt/b$ we get

$$E_y(\omega) = \frac{1}{4\pi\varepsilon_0} \frac{ze}{\sqrt{2\pi} bv} \int_{-\infty}^{\infty} \frac{e^{i\omega bu/\gamma v}}{(1+u^2)^{3/2}} du \qquad (33)$$

The integral in this equation is related to the *modified Bessel Function* K_1. We have the integral representation of modified Bessel Function $K_\nu(z)$ given by [See Handful of Mathematical Functions, Ed. By Abramowitz and Segun]:

$$K_\nu(z) = \frac{2^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} z^\nu} \int_0^\infty \frac{\cos(zt) dt}{(1+t^2)^{\nu+\frac{1}{2}}} \qquad (34)$$

In particular

$$\begin{aligned} K_1(z) &= \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi} z} \int_0^\infty \frac{\cos(zt) dt}{(1+t^2)^{\frac{3}{2}}} = \frac{1}{2z} \int_{-\infty}^{\infty} \frac{\cos(zt) dt}{(1+t^2)^{\frac{3}{2}}} \\ &= \frac{1}{2z} \int_{-\infty}^{\infty} \frac{e^{izt} dt}{(1+t^2)^{\frac{3}{2}}} \end{aligned} \qquad (35)$$

Similarly

$$K_0(z) = \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} \int_0^{\infty} \frac{\cos(zt)dt}{(1+t^2)^{\frac{1}{2}}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{izt}dt}{(1+t^2)^{\frac{1}{2}}}$$

On partial integration this becomes

$$K_0(z) = \frac{1}{2iz}\int_{-\infty}^{\infty} \frac{te^{izt}dt}{(1+t^2)^{\frac{3}{2}}} \tag{36}$$

On using equation (35), equation (33) becomes

$$E_y(\omega) = (\frac{2}{\pi})^{1/2} \frac{1}{4\pi\varepsilon_0} \frac{ze}{bv}[\frac{\omega b}{\gamma v} K_1(\frac{\omega b}{\gamma v})] \tag{37}$$

We do the calculation for the *x*-component in a similar fashion. On substituting for $E_x(t)$ from equation (1) into equation (31), we have

$$E_x(\omega) = -\frac{1}{4\pi\varepsilon_0}\frac{ze\gamma v}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{te^{i\omega t}}{(b^2+\gamma^2 v^2 t^2)^{3/2}}dt$$

The same change of variable, $u = \gamma vt/b$, and use of equation (36) for the integral yields in case

$$E_x(\omega) = -i(\frac{2}{\pi})^{1/2} \frac{1}{4\pi\varepsilon_0} \frac{ze}{b\gamma v}[\frac{\omega b}{\gamma v} K_0(\frac{\omega b}{\gamma v})] \tag{38}$$

The energy transfer to a harmonically bound charge can now be calculated explicitly. Substituting equations (37) and (38) for $E_y(\omega)$ and $E_x(\omega)$ respectively into equation (30), we obtain

$$\Delta W(b) = (\frac{1}{4\pi\varepsilon_0})^2 \frac{2z^2e^4}{mv^2}(\frac{1}{b^2})[\xi^2 K_1{}^2(\xi) + \frac{1}{\gamma^2}\xi^2 K_0{}^2(\xi)]$$

where

$$\xi = \frac{\omega_0 b}{\gamma v} = \frac{b}{b_{\max}}$$

The factor outside the square brackets is just the approximate result for energy loss, equation (3). The asymptotic forms of K_0 and K_1 for large and small argument are well known and are

(i) $For\,\xi \to 0$

$$\xi K_0(\xi) \to 0;$$
$$\xi K_1(\xi) \to 1$$

(ii) $For\,\xi \to \infty$

$$K_\nu(\xi) \to \sqrt{\frac{\pi}{2\xi}}e^{-\xi}$$, independent of ν.

Using these asymptotic forms we see that the term in the square brackets tends to 1 for $\xi << 1$, and to $(1+\frac{1}{\gamma^2})\frac{\pi}{2}\xi e^{-2\xi}$ for $\xi >> 1$. Since $\xi = \frac{b}{b_{\max}}$, we see that the energy transfer is essentially the approximate result given by equation (3) for $b << b_{\max}$. For $b >> b_{\max}$ it falls off exponentially to zero. This is as was expected from the qualitative argument given earlier.

Summary:

1. *An overview of the problem of energy loss by a charged particle in passing through matter is given.*
2. *Formula for energy transfer to a free atomic electron by a fast moving charged particle is derived.*
3. *Limitations of the formula in the small distance and large distance limits and the corresponding corrections are given.*
4. *Energy loss by the moving particle per unit distance is obtained.*
5. *Effect of the binding of the electron on the energy transfer is discussed and the formula for harmonically bound charge is derived.*